Abstract

This is an project report about some basic topics in Discrete Mathematics including some portions of combinatorics and number theory, which I studied under Dr. Srilakshmi Krishnamoorthy of Indian Institute of Technology, Madras (IIT-M) as the guide during the period of time from 25 May 2015 to 12 July 2015. I would like to thank Dr. Srilakshmi Krishnamoorthy by giving his valuable time to guide me.

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Chapter 1

Basic Combinatorics

The principle of addition:-
If a finite set S is a union of disjoint non-empty subsets $S_1, S_2, ..., S_n$, then $|S| = |S_1| + |S_2| + ... + |S_n|$. In other words if $S_1, S_2, ..., S_n$ is a partition of S then $|S| = |S_1| + |S_2| + ... + |S_n|$ holds.
If $E_1, E_2, ..., E_n$ are mutually exclusive events with E_i can happen in e_i ways, then E_1 or E_2 or ... or E_n can happen in $e_1 + e_2 + ... + e_n$ ways.

The principle of multiplication:- If $S_1, S_2, ..., S_n$ are non empty finite sets , then the no of elements in Cartesian product $S_1 \times S_2 \times ... \times S_n$ is the product $|S_1| \times |S_2| \times ... |S_n|$.

Number of k permutations of an n-set:-
A k permutation of an n-set is an ordered selection (without replacement of any elements) of k elements from an n-set. On the first hand we have n choices, once we have chosen the first element we have $n-1$ choice, similarly after we choose our $k-1$ th element we are left with $n-(k-1) = n-k+1$ choices. Let $S = \{a_1, a_2, ..., a_n\}$ be the set with n elements. Now let S_1 be the set corresponding to the possible elements for first choice. Then $S = S_1$ and $|S_1| = n$. Without loss of generality let a_n be the first element chosen. After this let S_2 be the set corresponding to the possible elements for the 2nd choice. $S_2 = \{a_1, a_2, ..., a_{n-1}\}$, $|S_2| = n-1$. Again without loss of generality let a_{n-1} be the second element chosen. S_3 be the set corresponding to the possible element for the 3rd choice. Proceeding in similar fashion for kth choice the set corresponding will be $S_k = \{a_1, ..., a_{n-k+1}\}$ and $|S_k| = n-k+1$. Let P be a k permutation of n-set i.e. $P = (p_1, p_2, ..., p_k)$ where $p_i \in S_i$. The set of all such P will be the Cartesian product of $S_1, S_2, ..., S_k$ i.e. $S_1 \times S_2 \times ... \times S_k$ of and by principle of multiplication $|S_1 \times S_2 \times ... \times S_k| = |S_1| \times |S_2| \times ... \times |S_k| = n(n-1)...(n-k+1)$. Hence the no of k permutations of an n-set is given by $n \times (n-1)...(n-k+1)$.We denoted it by ${}^nP_k$.

Let us see what happens to ${}^nP_k$ when k varies over natural numbers. Clearly as long as S_k is non empty ${}^nP_k$ is non zero. By similar argument in the proof we will have S_k non-empty if $n-k+1 \geq 1$ i.e. $n \geq k$. In particular for $n = k$, ${}^nP_n = n \times (n-1)...3 \times 2 \times 1$. We denote this by $n!$ i.e. ${}^nP_n = n!$. For $n < k$, ${}^nP_k = 0$ vacuously as it is not possible to choose more than n elements from an

n-set without replacement of any element.

Number of k combinations of an n-set:-
A k combination of an n set is an unordered selection. We denote it by ${}^nC_k$. We wish to have an explicit formula for ${}^nC_k$ like we have for ${}^nP_k$. For this we make a two way of counting. Number of k permutations of n-set is ${}^nP_k$. In other way we can first select k-subset in ${}^nC_k$ ways (as a k combination of an n-set is just selecting k elements from n-set with out considering any order in which these k elements were chosen) and order those elements in ${}^kP_k$ways (clearly after we select a k combination of an n set we are left with a k-set in unordered manner. Now we have to order this k-subset which is same as taking k permutation of a k-set). Now let $\mathbf{C}(n,k)$ be the set of all k combinations of an n-set, $\mathbf{P}(n,k)$ and $\mathbf{P}(k,k)$ be the set of all k permutations of a n-set and a k-set respectively. It is clear that $|\mathbf{P}(n,k)| = |\mathbf{C}(n,k)| \times |\mathbf{P}(k,k)|$ (As each k permutation of an n-set is uniquely determined by a unique couple of k combination of the same n-set and k permutation of the chosen k-set). Hence ${}^nP_k = {}^nC_k \times {}^kP_k$ i.e.
${}^nC_k = \frac{{}^nP_k}{{}^kP_k} = \frac{n(n-1)...(n-k+1)}{k!}$

We will use some further notation for convenience. We write $[n]_k$ to denote ${}^nP_k = n(n-1)...(n-k+1)$ and a call it a falling factorial. By convention $0! = 1$
We then have ${}^nC_k = \frac{[n]_k}{k!} = \frac{[n]_k(n-k)!}{k!(n-k)!} = \frac{n!}{k!(n-k)!}$

Rule of bijection
Let A be B be two finite set. If there exists a bijective function $f : A \to B$, then $|A| = |B|$.
Let T be a n-set. We wish to show that no of subsets of T is 2^n. With out loss of generality we can assume $T = \{1, 2, ..., n\}$. Let B_T be set of all binary sequence of length n. i.e. for $b_S \in B_S$, $b_S = b_{S_1}b_{S_2}...b_{S_n}$ such that $b_{S_i} = 0$ or 1 for $1 \leq i \leq n$. Clearly $B_S = B_{S_1} \times B_{S_2} \times ... \times B_{S_n}$ with $B_{S_1} = B_{S_2} = ... = B_{S_n} = \{0, 1\}$. Hence by principle of multiplication $|B_S| = 2^n$.
Let $P(T)$ be the power set of T i.e. the set that contains all the subsets of T.
Consider $f : P(T) \to B_S$ such that for $S \in P(T)$, $b_S = b_{S_1}b_{S_2}...b_{S_n} = f(S)$
$b_{S_i} = 0$ if $i \notin S$,
1 if $i \in S$ where $1 \leq i \leq n$. (in particular for $T = \{1, 2, 3, 4, 5, 6\}$ and $S = \{1, 2, 5\}$ $b_S = f(S) = b_{S_1}b_{S_2}b_{S_3}b_{S_4}b_{S_5} = 110010$)
Clearly f is a bijection as every pre-image of $b_S \in B_S$ is a unique subset of T. Hence $|P(T)| = 2^n$.

Result 1:-
Let m and n be positive integers. Then there exists a bijection between any two of the following sets.
(i) The set of all the functions from an n-set to an m-set.
(ii) The set of words of length n on an alphabet of m letters.
(iii) The set of n-tuples with entries from an m-set.
(iv)The set consisting of all the ways of **distributing** n **distinct** objects into m **distinct** boxes.
In each case the cardinality of the set given is m^n.

Proof:- Let $T = \{1, 2, ..., n\}$ be the n-set and $S = \{s_1, s_2, ...s_m\}$ be the m-

set. Let $f : T \to S$ be any given function. If we allow this f map to the n-tuple $(f(1), f(2), ..., f(n))$ of length n. Clearly $f(i)$ $(1 \le i \le n)$ can take any value from S itself which is a m-set. Similarly for a given m-set B and a given n-tuple $(a_1, a_2, ...a_n)$ such that $a_i \in B$, we map this n-tuple to a function $f : T \to B$ such that $f(i) = a_i$. This sets a bijection between set in (i) and (ii). It is not difficult to give a bijection between (ii) and (iii). For (i) and (iv), given a function $f : T \to S$ such that $f(i) = j$, put ith object in B_jth box. Similarly we can also state the reverse direction. This completes the bijection between the given sets. Which from rule of bijection follows that each set have the same cardinality. Cardinality set of n-tuples with entries from m-set is m^n follows from principle of multiplication.

Result 2:-
Let m and n be positive integers. Then Then there exists a bijection between any two of the following sets.
(i) The set of all the **injective** functions from an n-set to an m-set .
(ii) The set of words of length n on an alphabet of m letters with the condition that the word consists of **distinct** letters.
(iii) The set of n-tuples with **distinct** entries from an m-set.
(iv)The set consisting of all the ways of **distributing** n **distinct** objects into m **distinct** boxes with the condition that no box holds **more than one** object.

In each case the cardinality of the set given is ${}^mP_n = [m]_n$.

Proof :- Let $T = \{1, 2, ..., n\}$ be the n-set and $S = \{s_1, s_2, ...s_m\}$ be the m-set. Let $f : T \to S$ be any given injective function. Clearly $f(i)$ $(1 \le i \le n)$ and $f(i) \ne f(j)$ $\forall i \ne j$. If we allow this f map to the n-tuple $(f(1), f(2), ..., f(n))$ of length n can take any value from S itself which is a m-set. Also the entries in the n-tuple will be distinct as $f(i) \ne f(j)$ $\forall i \ne j$. Similarly for a given m-set B and a given n-tuple $(a_1, a_2, ...a_n)$ such that $a_i \in B$ and $)a_i \ne a_j$ $\forall i \ne j$, we map this n-tuple to a function $f : T \to B$ such that $f(i) = a_i$. $a_i \ne a_j$ $\forall i \ne j$ ensures that f will be injective. This sets a bijection between set in (i) and (ii). It is not difficult to give a bijection between (ii) and (iii). For (i) and (iv), given a function $f : T \to S$ such that $f(i) = j$ and $f(i) \ne f(j)$ $\forall i \ne j$, put ith object in B_jth box. The constraint $f(i) \ne f(j)$ $\forall i \ne j$ takes care about not any box will get more than one object. Similarly we can also state the reverse direction. This completes the bijection between the given sets. Which from rule of bijection follows that each set have the same cardinality. Cardinality set of n-tuples with distinct entries from m-set is ${}^mP_n$ which follows from $k - permutatationsofan - set$ given previously.

Result 3:-
Let M be a multi-set containing r distinct objects each with infinite multiplicities (to ensure that r may repeat in the set as many times). Then the total number of d-permutations of M is r^d.

Proof :- The total no of doing the same is counting the no of words of length d on an alphabet that consists of r distinct letters. Now as each letter is allowed to repeat any number of times we never fall short of any letter. Hence by Result $1(ii)$ the required number is r^d.

Remark 1:-
While defining binomial coefficient ${}^{n}C_k$ we actually count number of ways of putting n distinct objects into two distinct boxes labelled B_1 and B_2 such that the first box contains k objects and the second box contains $n-k$ objects. Generalizing this would be a multinomial coefficient i.e. the no of ways of putting n distinct objects in r distinct boxes $B_1, B_2, ..., B_r$ such that the i-th box B_i holds n_i objects is called a multinomial coefficient and is denoted by ${}^{n}P_{n_1,n_2,...,n_r}$ necessarily then $n_1 + n_2 + ... + n_r = n$.

Result 4:-
Let S be a n-set and suppose the n elements/objects in S are to be put in r distinct boxes $B_1, B_2, ..., B_r$ such that the i-th box B_i contains n_i objects with $n_1 + n_2 + ...n_r = n$. The number of ways of doing this is equal to ${}^{n}P_{n_1,n_2,...,n_r} = \frac{n!}{n_1!n_2!...n_r!}$

Proof :- With out loss of generality we can say firstly we select n_1 objects of n-set and put them in B_1 box, which can be done in ${}^{n}C_{n_1}$ ways. After this we are left with $n - n_1$ objects out of which we will select n_2 boxes put it into B_2, which can be done in ${}^{n-n_1}C_{n_2}$ ways. Proceeding in similar manner finitely upto $r-2$ more times and from principle of multiplication it follows that number of ways of doing it will be
${}^{n}C_{n_1} \times {}^{n-n_1}C_{n_2} \times {}^{n-n_1-n_2}C_{n_3} \times ... \times {}^{n-(n_1+...+n_{r-1})}C_{n_r} = \frac{n!}{n_1!(n-n_1)!} \times \frac{(n-n_1)!}{n_2!(n-n_1-n_2)!} \times {}^{n-n_1-n_2}C_{n_3} \times ... \times {}^{n-(n_1+...+n_{r-1})}C_{n_r} = \frac{n!}{n_1!n_2!(n-n_1-n_2)!} \times {}^{n-n_1-n_2}C_{n_3} \times ... \times {}^{n-(n_1+...+n_{r-1})}C_{n_r} = \frac{n!}{n_1!n_2!...n_r!}$ (by continuing on similar fashion)

Corollary :-
Let M be a multi-set consisting of r distinct objects $x_1, x_2, ..., x_r$ such that the i-th object x_i has multiplicity n_i. Let $n = n_1 + n_2 + ... + n_r$. Then the total number of n-permutations of M is ${}^{n}P_{n_1,n_2,...,n_r} = \frac{n!}{n_1!n_2!...n_r!}$

Proof :- We set up a bijection between the required set of all ways of elements putting the elements of S in r boxes and all the $n$permutations of the multi-set M. First of all we number the elements of S from 1 to n. If element i is put in the box B_j, then make a n-permutation in which ith place is occupied by x_j. Conversely, given an n-permutation of M. if ith place is occupied by x_j then put the element i in the B_j box. The result follows from rule of bijection.

Result 5:-
Let M be a multi-set with r distinct objects $x_1, x_2, ..., x_r$ each with infinite multiplicity. The number of k-combinations of M is ${}^{k+r-1}C_k = {}^{k+r-1}C_{r-1}$.

Proof :-Every k-combination is uniquely determined by a sequence $b_1, b_2, ..., b_r$ where b_i $\forall i$ are non-negative integer and $b_1 + b_2 + ...b_r = k$. For each given k-combination of M make a binary sequence of length $k + r - 1$ as follows. At the starting write b_1 zeros and follow this by a 1, then write b_2 zeros and then write 1 again and so on. There will be a 1 separating b_{r-1} zeros and b_r zeros. Thus the binary sequence will consist of exactly $b_1 + b_2 + ... + b_r = k$ zeros and $r - 1$ ones. Conversely for a given binary sequence of length $k + r - 1$ consisting of exactly k zeros and $r-1$ ones, we read the no of zeros from left to the first one and call it b_1, then we call b_2 as the no of zeros between the first one and the last one and so on. Finally the no of zeros to the right of the last one is b_r. Since the no of binary sequence of length $k+r-1$ with exactly k zeros is equal to ${}^{k+r-1}C_k = {}^{k+r-1}C_{r-1}$(from problem 1(i)) result follows from rule of bijection.

Alternative proof :- Consider a $(k+r-1)$-set S with elements of S numbered from 1 to $k+r-1$. Every k-combination is uniquely determined by a sequence $b_1, b_2, ..., b_r$ where b_i $\forall i$ are non-negative integer and $b_1+b_2+...b_r = k$. For given such k-combination of M we remove the element next to b_1, then we remove the element next to b_1+b_2 and so on. We remove elements upto the next element of $b_1+b_2+...+b_{r-1}$ (including the next element of $b_1+b_2+...+b_{r-1}$. After this removal of $r-1$ elements we are left with a k-subset of S. Conversely for a given such k-subset of S, we number the elements of S and following to this numbering we see which numbered elements appear in the k-subset. We assign b_1 to the no of numbered elements in S which are in between the first numbered and the second numbered element of the k-subset, then assign b_2 to the no of numbered elements in S which are in between the second and third numbered element of the k-subset and so on. Finally we assign b_r to the no of numbered elements in S after kth numbered element in k-subset. Number of ways of choosing k-subsets of a $k+r-1$ subset is ${}^{k+r-1}C_k$, result follows from rule of bijection.

Corollary :-
The number of ways of putting k **identical** objects into r **distinct** boxes with each box containing at least one object is ${}^{k-1}C_{r-1}$

Proof :- The scenario is quite similar to the scenario in Result 5 except the constraint that each box contains atleast one. What we do is we put exactly one object into each box. After this we are left with $k-r$ objects as r boxes will contain exactly r objects with each containing exactly one. This problem now reduces to Result 5, however here we have $k-r$ objects here with r distinct boxes which can be put in ${}^{(k-r)+r-1}C_{r-1} = {}^{k-1}C_{r-1}$ ways.

Chapter 2

Stirling numbers

Definition 1:-
Let n, k be positive integers with $n \leq k$. The Stirling number of second kind $S(n,k)$ is the total number of partitions of an n-set into k disjoint, non-empty and unordered subsets.

We take the n-set to be $\{1, 2, ..., n\}$, denote this set by $[n]$. Let us take one example and see. For $n = 5$ and $k = 1$ there is only one way to have partition of [5] into 1 disjoint, non-empty and ordered set. Which is $[n] = [n]$ itself. $S(5,1) = 1$
For $n = 5$ and $k = 2$. Let us explicitly write down the possible partitions. $[n] == [1] \cup \{2,3,4,5\} = \{2\} \cup \{1,3,4,5\} = \{3\} \cup \{1,2,4,5\} = \{4\} \cup \{1,2,3,5\} = \{5\} \cup [4] = [2] \cup \{3,4,5\} = \{1,3\} \cup \{2,4,5\} = \{1,4\} \cup \{2,3,5\} = \{1,5\} \cup \{2,3,4\} = \{2,3\} \cup \{1,4,5\} = \{2,4\} \cup \{1,3,5\} = \{2,5\} \cup \{1,3,4\} = \{3,4\} \cup \{1,2,5\} = \{3,5\} \cup \{1,2,4\} = \{4,5\} \cup [3]$
Note that we will not count $\{3,4,5\} \cup [2]$ as distinct as it already corresponds to $[2] \cup \{3,4,5\}$ for we need to count unordered subsets. There are 15 such possibilities. So $S(5,2) = 15$.We can do this in this way. In order to get, suppose P_1 and P_2 be two partition (will not use unordered, disjoint, non-empty from now for convenience) of [5] . There are two case of getting partitions. One is when partitions are such that $|P_1| = 1$ which immediately imply $|P_2| = 2$, another one is $|P_1| = 2$ which imply $|P_2| = 3$. Number of ways of getting partition such that $|P_1| = 1$ is ${}^5C_1$ and getting partitions such that $|P_1| = 2$ is ${}^5C_2$. So the total number of ways of getting two partition is $S(5,2) = {}^5C_1 + {}^5C_2 = 15$.

Let us see what happens when $n = 6$ and $k = 2$. There are three cases of getting two partitions. And the cases are when $|P_1| = 1$, $|P_1| = 2$ and $|P_1| = 3$. For $|P_1| = 1$ and $|P_1| = 2$ no of ways are ${}^6C_1$ and ${}^6C_2$ respectively. In case of $|P_1| = 3$ number of ways are $\frac{{}^6C_3}{2}$ (reason for dividing 2 as follows. For $|P_1| = 3$ ${}^6C_3$ chooses any subset with cardinality 3 so that [3] and $\{4,5,6\}$ to be counted different, however we can not do that as $[3] \cup \{4,5,6\}$ and $\{4,5,6\} \cup [3]$ are to be counted once). $S(6,2) = {}^6C_1 + {}^6C_2 + \frac{{}^6C_3}{2} = 6 + 15 + 10 = 31$

For $n = 6$ and $k = 3$. Let P_1, P_2 and P_3 be three partitions of [6]. There are three cases of getting such. Three cases are as follows when $|P_1| = |P_2| = 1$,

$|P_1| = 1, |P_2| = 2$, $|P_1| = |P_2| = 2$. For $|P_1| = |P_2| = 1$. For $|P_1| = |P_2| = 1$ no of ways are $\frac{^6C_{1,1,4}}{2}$ (In order to have $|P_1| = |P_2| = 1$, $|P_3| = 4$ we have to look at multinomial coefficient $^6C_{1,1,4}$. However ordered selection of P_1, P_2 and P_3 is included in this multinomial coefficient. For instance $\{1\}, \{2\}, \{3,4,5,6\}$ and $\{2\}, \{1\}, \{3,4,5,6\}$ are counted as different whereas they should be counted as once. Hence for each such pair(where order of P_1 and P_2 were taken) is to be counted once which justifies division by 2). For second case when $|P_1| = 1, |P_2| = 2$ and $|P_3| = 3$ no of ways of getting partition is $^6C_{1,2,3}$ (Note that there will not be any kind of division by a number (greater than 1) required as before. As for each choice for P_1, P_2 and P_3 it corresponds to a unique partition which in turn is due to the fact $|P_1| \neq |P_2| \neq |P_3|$). For $|P_1| = |P_2| = |P_3| = 2$ no of ways will be $\frac{^6C_{2,2,2}}{6}$ (In particular $([2], \{3,4\}, \{5,6\})$, $([2], \{5,6\}, \{3,4\})$, $(\{3,4\}, [2], \{5,6\})$, $(\{3,4\}, \{5,6\}, [2])$, $(\{5,6\}, [2], \{3,4\})$ and $(\{5,6\}, \{3,4\}, [2])$ are counted as different whereas for partition it has to be counted once.This justify division by 6). Hence $S(6,3) = \frac{^6C_{1,1,4}}{2} + {^6C_{1,2,3}} + \frac{^6C_{2,2,2}}{6} = 15 + 60 + 15 = 90$.

Remark 1:-
Finding Stirling number $S(n,k)$ an algorithm can be written as follows.
(i) Find no of cases such that $|P_1| + |P_2| + ... + |P_k| = n$ with distinct case have usual meaning viz. two cases are different iff at least one of $|P_i|$ where $1 \leq i \leq k$ is different.
(ii) For each case find no of ways of partition.
(iii) Add all the no of ways of getting partition corresponding to each case.

Lemma 1:-
No of ways of having partition for $|P_1|, |P_2|, ..., |P_k|$ is $\frac{^nP_{|P_1|,|P_2|,...,|P_k|}}{t}$, where t is as follows. Let $|P_{i_1^1}| = |P_{i_2^1}| = ... = |P_{i_{r_1}^1}|$ with $1 \leq i_1^1 < i_2^1 < ... < i_{r_1}^1 \leq k$, $|P_{i_1^2}| = |P_{i_2^2}| = ... = |P_{i_{r_2}^2}|$ with $1 \leq i_1^1 < i_2^1 < ... < i_{r_2}^2 \leq k$,..., $|P_{i_1^s}| = |P_{i_2^s}| = ... = |P_{i_{r_s}^s}|$ with $1 \leq i_1^t < i_2^t < ... < i_{r_s}^s \leq k$ and $r_1 + r_2 + ... r_s = k$ then $t = r_1! r_2! ... r_s!$.

Note that upper index of i is used to classify the cardinality of k partitions into s components such that each component correspond to a unique number and each jth component has r_j number of partitions corresponding to that unique number.

Proof :-No of ways of having **ordered** partitions for $|P_1|, |P_2|, ..., |P_k|$ is equal to putting n objects in k boxes with ith box receiving $|P_i|$ number of objects. Hence the required number will be $^nP_{|P_1|,|P_2|,...,|P_k|}$. However we have included ordered partition, which needs to be taken care of. Now Consider a fixed jth upper index of i. For this jth upper index $|P_{i_1^j}| = |P_{i_2^j}| = ... = |P_{i_{r_j}^j}|$ with $1 \leq i_1^j < i_2^j < ... < i_{r_j}^j \leq k$ i.e. there are r_j number of partitions out of k which have the same cardinality. It is quite clear that $P_{i_1^j} \neq P_{i_2^j} \neq ... \neq P_{i_{r_j}^j}$ which implies that ordering of these r_j partitions has already been considered in our counted ordered partitions $^nP_{|P_1|,|P_2|,...,|P_k|}$. But these ordering of r_j partitions which can be done in $r_j!$ ways (As all sets are distinct) which correspond to only one way when we take the unordered partition, meaning we have to divide $r_j!$ to the no of ways of ordered partitions in order to get unordered partition which is required. As this upper index j can take any value starting from 1 upto s we have to divide $r_j!$ for each j. Hence the desired number is $\frac{{}^nP_{|P_1|,|P_2|,...,|P_k|}}{t} = \frac{n!}{n_1!n_2!...n_k!r_1!r_2!...r_s!}$

Let us see with an example. Consider the third case while calculating $S(6,3)$. For third case $|P_1| = |P_2| = |P_3| = 2$ i.e. if we see this case from lemma 1 point of view $|P_{i_1^1}| = |P_{i_2^1}| = |P_{i_{r_1}^1}|$ where $s = 1$ and $r_1 = 3$. $t = r_1! = 3! = 6$. Consider the second case of the same when $|P_1| = |P_{i_1^1}| = 1, |P_2| = |P_{i_1^2}| = 2$ and $|P_3| = |P_{i_1^3}| = 3$. So $s = 3$, $r_1 = r_2 = r_3 = 1$, $t = 1! \times 1! \times 1! = 1$. For the first case $|P_1| = |P_2| = 1$, $|P_3| = 4$, $s = 2$, $r_1 = 2$, $r_2 = 1$, $t = 2! \times 1! = 2$.

Theorem 1:-
$S(n,k) = \sum \frac{{}^nP_{|P_1|,|P_2|,...,|P_k|}}{t} = \sum \frac{n!}{n_1!n_2!...n_k!r_1!r_2!...r_s!}$ where sum is taken over all unordered k-tuples $(|P_1|, |P_2|, ..., |P_k|)$ such that $|P_1| + |P_2| + ... + |P_k| = n$.

Proof :- Proof follows from Remark 1 and Lemma 1.

Theorem 2:-
Let n and k be positive integers with $n \geq k$. Then
$S(n,k) = S(n-1,k-1) + kS(n-1,k)$

Proof :- Let X be a n-set and $X = \cup_{i=1}^{k} X_i$ be a given partition of X. For $x \in X$, define $Y = X - \{x\}$ For given partition $\{X_1, X_2, ..., X_k\}$ gives a natural partition of $Y = \cup_{i=1}^{k} Y_i$. With out loss of generality we assume that $x \in X_k$. Now we have two cases one is when $X_k = \{x\}$ and another is when $X_k \neq \{x\}$. When $X_k = \{x\}$, Y_k is empty, which imply $Y = \cup_{i=1}^{k-1} Y_i$, i.e. we get a $k-1$ partition of Y. This process is also reversible. For a given partition $\{Y_1, Y_2, ..., Y_{k-1}\}$, we take $X_i = Y_i$ for $i = 1, 2, ..., k-1$ and $X_k = \{x\}$. Clearly this case can arise in $S(n-1, k-1)$ ways.
For the second case $X_k \neq \{x\}$ for a given partition $\{X_1, X_2, ..., X_k\}$ corresponding partition will be $\{Y_1, Y_2, ..., Y_k\}$ with each Y_i non empty. This can be done in $S(n-1, k)$ ways. However given any such partition $\{Y_1, Y_2, ..., Y_k\}$ we can choose to insert into any one of the Y_i and depending upon the which Y_i we choose, we get a different partition of X. That is the correspondence between the set of partitions of Y into k non-empty unordered disjoint subsets and the set of partitions of X into k non-empty unordered disjoint subset with the subset containing x having at least two elements is $1 : k$. For the second case no of partitions will be $kS(n-1,k)$. Hence $S(n,k) = S(n-1,k-1) + kS(n-1,k)$.

This recurrence relation is quite helpful. For instance in the example given in the beginning to evaluate $S(6,2)$, $S(6,2) = S(5,1) + 2S(5,2) = 1 + 2 \times 15 = 31$. Similarly $S(7,3) = S(6,2) + 3S(6,3) = 31 + 3 \times 90 = 301$.

Theorem 3:-
The number of surjective (onto) functions from an n-set to k-set($n \geq k$) is equal to $k!S(n,k)$.

Proof :- Let $A = \{a_1, a_2, ..., a_n\}$ be the n-set and $B = \{b_1, b_2, ..., b_k\}$ be the k-set. A function $f : A \to B$ will be surjective if $\forall\ j \in \{1, 2, ...k\}\ \exists\ i \in \{1, 2, ..., n\}$ such that $f(a_i) = b_j$. We wish to calculate all such possible f. Let $\{A_1, A_2, ..., A_k\}$ be a given k partition of A. For this given partition we assign a function $g : P(A) \to B$ such that $g(A_i) = b_j$ for all $1 \le i, j \le k$. For a given partition A_i is fixed. However assigning these A_i to b_j by the function g has no constraint on j except $1 \le j \le k$. To be precise $g(A_1)$ has k many choices viz. $b_1, b_2, ..., b_k$. Once we assign one of these to $g(A_1)$, we have $k-1$ many choices. Similarly we proceed finitely. So the no of such $g : P(A) \to B$ will be $k!$. Number of k partitions of A is $S(n, k)$. Hence number of such g will be $k!S(n, k)$ (quite evident from rule of multiplication). Now for a given $g : P(A) \to B$ we can find $f : A \to B$ just by sending all the elements for each partition A_i to the corresponding given b_j (as g is given). In reverse direction for a given $f : A \to B$ we for each b_j we consider $f^{-1}(b_j)$ (where $f^{-1}(b_j) = \{a_i \in A : f(a_i) = b_j\}$). Clearly $\cup_{j=1}^{k} f^{-1}(b_j) = A$ and $g : P(A) \to B$ will be corresponding to the given $f : A \to B$ with $g(f^{-1}(b_j)) = b_j$. This gives a bijection and completes the proof.

Note :- It is quite clear that for $n \le k$ there will not be any onto function from an n-set to a k-set.

Corollary :-
The number of ways of **putting** n balls of **distinct** colours into k **distinct** boxes with each boxes containing at least one ball is equal to $k!S(n, k)$.

Proof :- All we need is to give a bijection. For given an n-set A, k-set B and a $f : A \to B$ put the ith coloured ball into jth box if $f(a_i) = b_j$. Similarly we can give for reverse direction.
Note :- In an informal sense n distinct coloured balls ensures we have a n-set, k distinct boxes ensures a k-set while each boxes containing atleast one ball takes care of surjective part.

Theorem 4:-
For any positive integer m and n we have
$m^n = \sum_{k=1}^{min(m,n)} {}^{m}C_k k!S(n, k) = \sum_{k=1}^{min(m,n)} {}^{m}P_k S(n, k) = \sum_{k=1}^{min(m,n)} [m]_k S(n, k)$

Proof :- Let A and B be n-set and m-set respectively. Left most side counts no of functions from A to B. For $n \ge m$ every function from A to B has a unique range C which is some k-subset of B where $1 \le k \le m = min(m, n)$. C being k-subset of B can be chosen in ${}^{m}C_k$ ways and number of functions such that C is the range is $k!S(n, k)$, result follows from summing over k. For $n < m$ every function from A to B has a unique range C which is again some k-subset of B where $1 \le k \le n = min(m, n)$ (as for $n < k < m$ there will not be any function due to the fact that for any function cardinality of range(finite) can not exceed the cardinality of domain(finite)). Proof is same from this point. Right most side follows from definition itself.

Corollary :-
Let n be a positive integer. Then the following formal polynomial identity holds
$x^n = \sum_{k=1}^{n} S(n, k)[x]_k$

Proof :-Since n is fixed both the sides are of the degree $\le n$ implying polynomial

$(x^n - \sum_{k=1}^{n} S(n,k)[x]_k)$ has a degree n. The polynomial $x^n - \sum_{k=1}^{n} S(n,k)[x]_k$ has its roots at $x = 1, 2, .., n$ as for such choices $x \leq n$ and $x^n = \sum_{k=1}^{min(x,n)} S(n,k)[x]_k$ holds by theorem 3. Also $x = 0$ trivial root of the polynomial. Hence we have a n degree polynomial having $n+1$ roots meaning the polynomial has to be identically 0 which completes the proof.